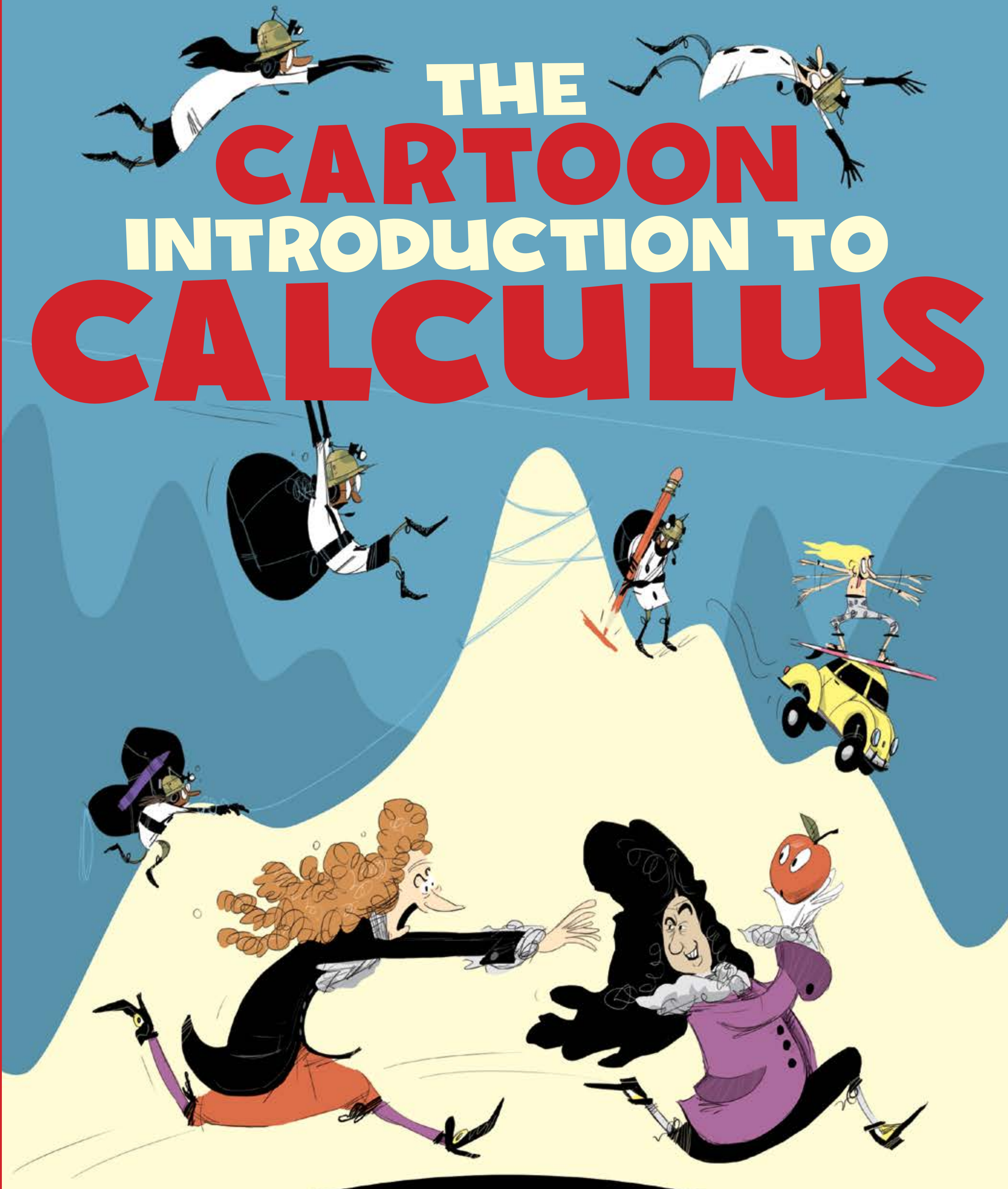
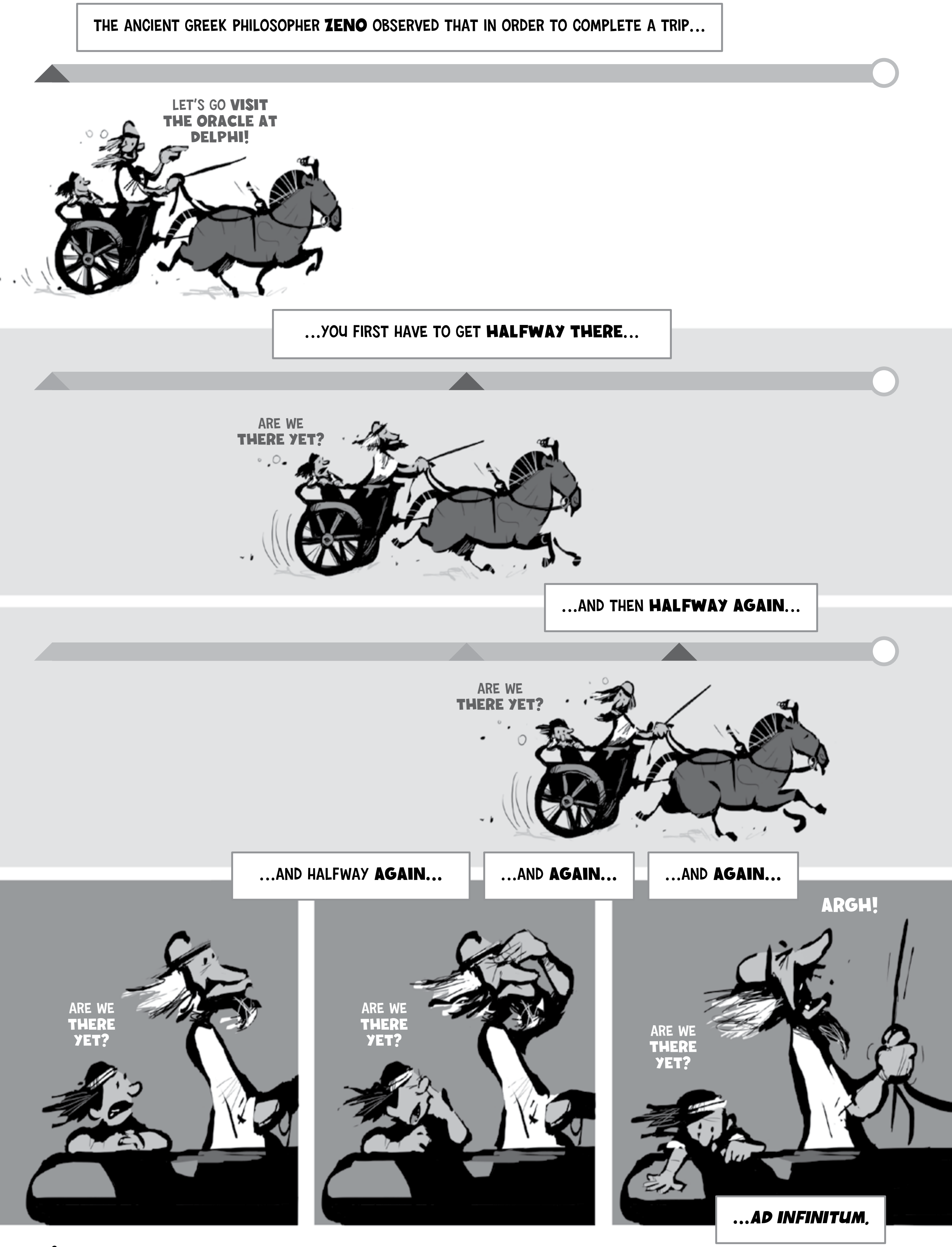
THE CARTOON INTRODUCTION TO CALCULUS

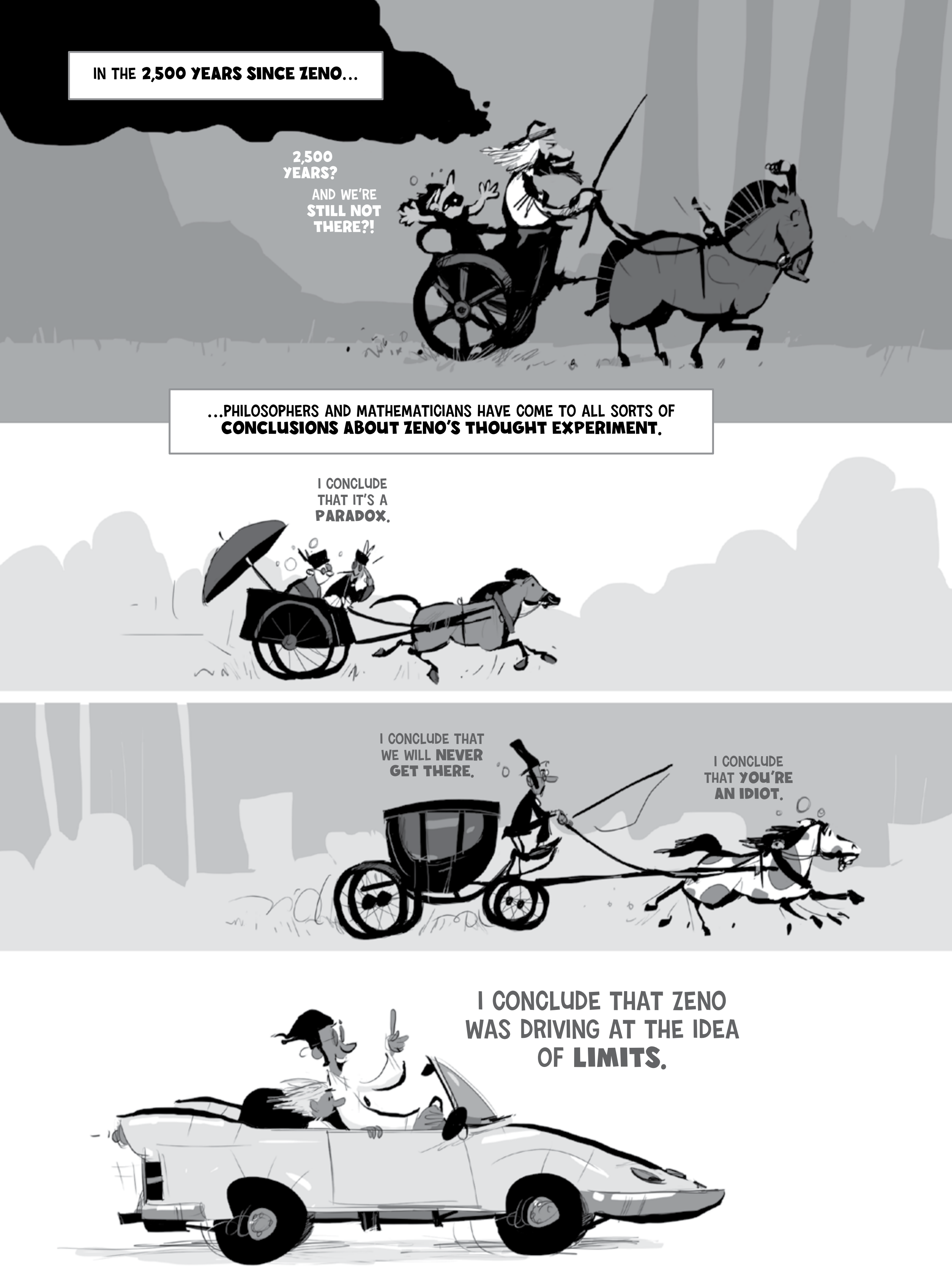


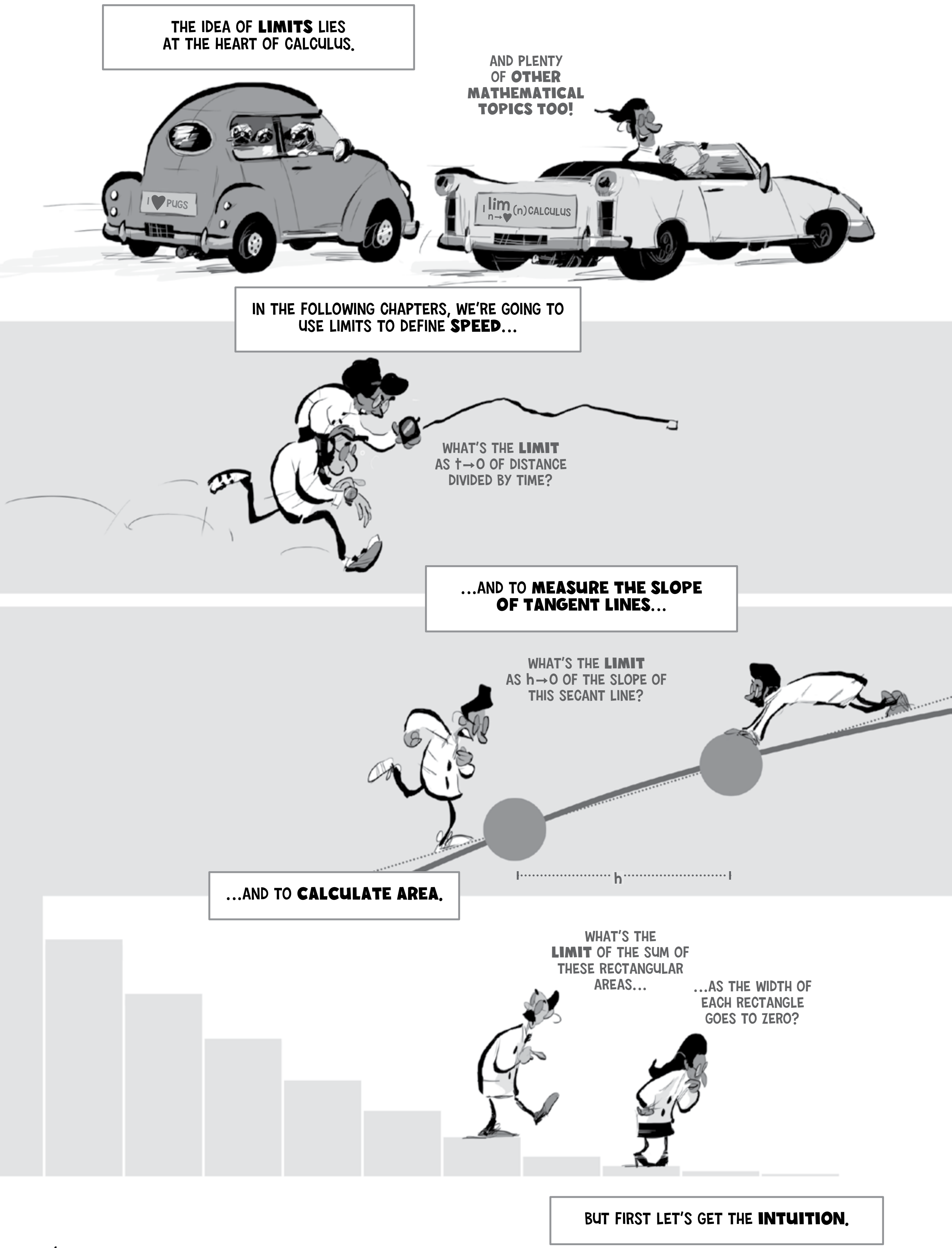
BY GRADY KLEIN AND
YORAM BAUMAN, Ph.D.
THE WORLD'S FIRST AND ONLY STAND-UP ECONOMIST

CONTENTS

CHAPTER 5
LIMITS


THE ANCIENT GREEK PHILOSOPHER ZENO OBSERVED THAT IN ORDER TO COMPLETE A TRIP...
LET'S GO VISIT THE ORACLE AT DELPHI!
...YOU FIRST HAVE TO GET HALFWAY THERE...
ARE WE THERE YET?
...AND THEN HALFWAY AGAIN...
ARE WE THERE YET?
...AND HALFWAY AGAIN...
ARE WE THERE YET?
...AND AGAIN...
ARE WE THERE YET?
...AND AGAIN...
ARGH!
ARE WE THERE YET?
...AD INFINITUM.

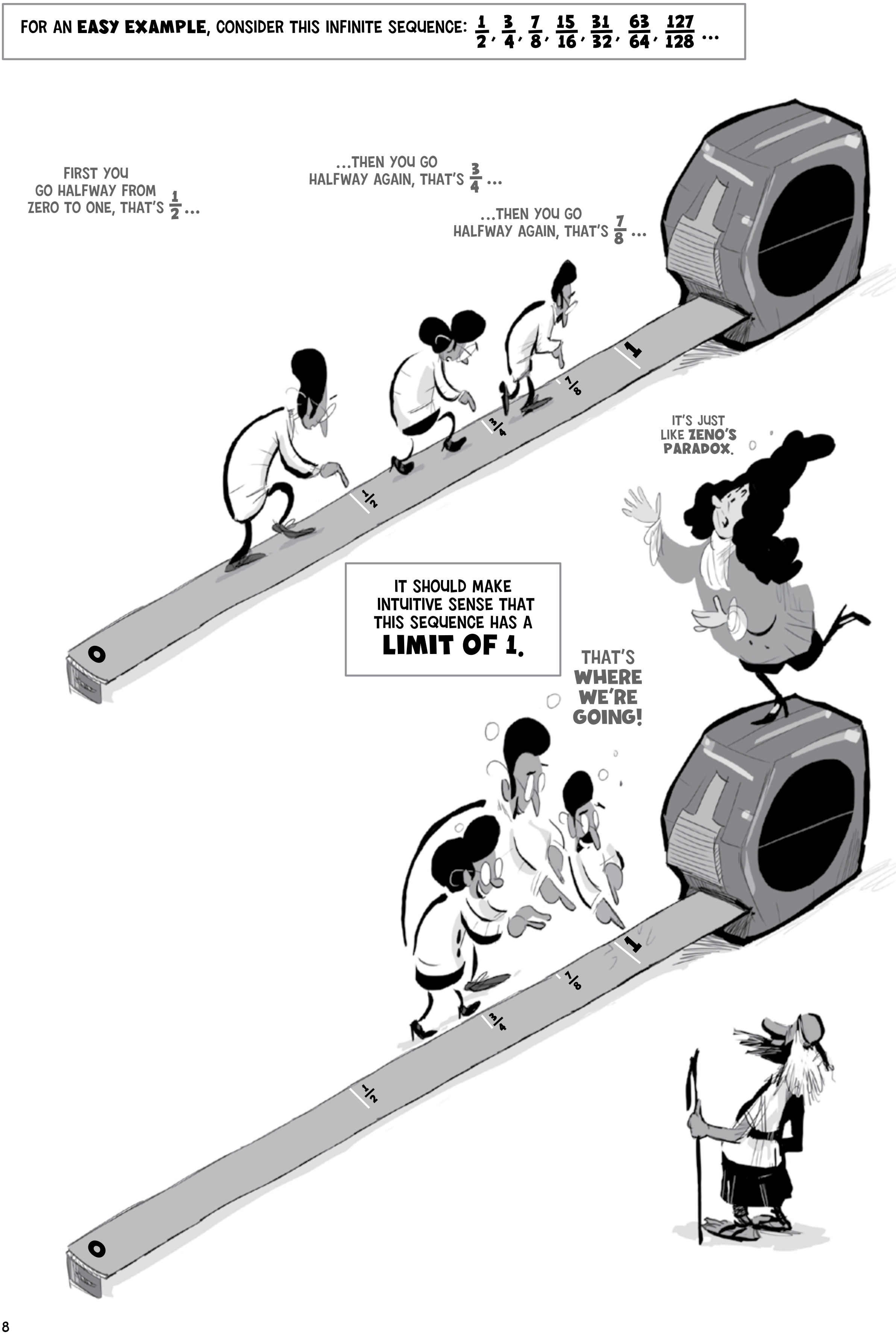

IN THE 2,500 YEARS SINCE ZENO...
2,500 YEARS?
AND WE'RE STILL NOT THERE?!
...PHILOSOPHERS AND MATHEMATICIANS HAVE COME TO ALL SORTS OF CONCLUSIONS ABOUT ZENO'S THOUGHT EXPERIMENT.
I CONCLUDE THAT IT'S A PARADOX.
I CONCLUDE THAT WE WILL NEVER GET THERE.
I CONCLUDE THAT YOU'RE AN IDIOT.
I CONCLUDE THAT ZENO WAS DRIVING AT THE IDEA OF LIMITS.

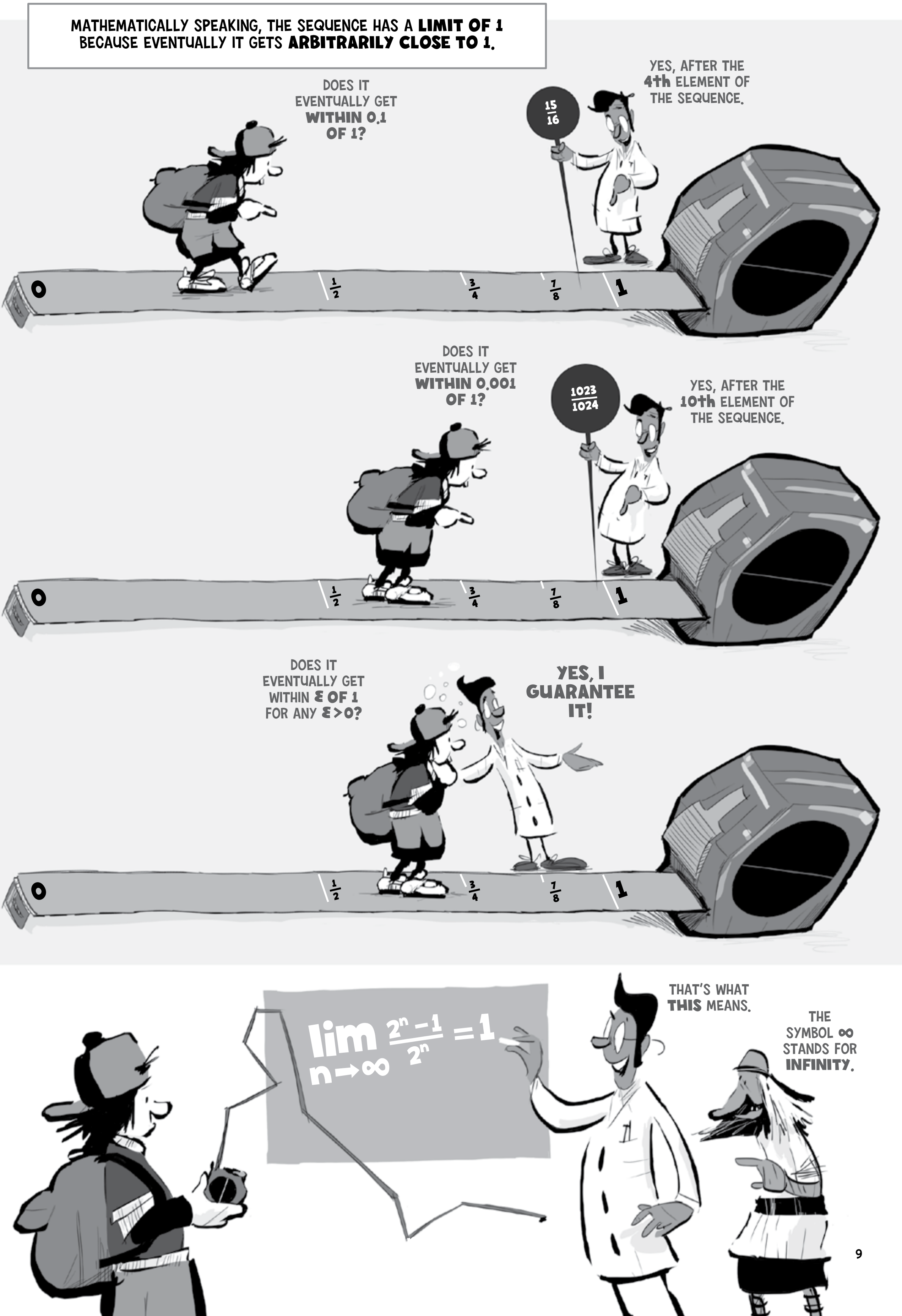

THE IDEA OF LIMITS LIES AT THE HEART OF CALCULUS.
AND PLENTY OF OTHER MATHEMATICAL TOPICS TOO!
I ♥ PUGS
lim n→♥ (n) CALCULUS
IN THE FOLLOWING CHAPTERS, WE'RE GOING TO USE LIMITS TO DEFINE SPEED...
WHAT'S THE LIMIT AS t→0 OF DISTANCE DIVIDED BY TIME?
...AND TO MEASURE THE SLOPE OF TANGENT LINES...
WHAT'S THE LIMIT AS h→0 OF THE SLOPE OF THIS SECANT LINE?
h
...AND TO CALCULATE AREA.
WHAT'S THE LIMIT OF THE SUM OF THESE RECTANGULAR AREAS...
...AS THE WIDTH OF EACH RECTANGLE GOES TO ZERO?
BUT FIRST LET'S GET THE INTUITION.

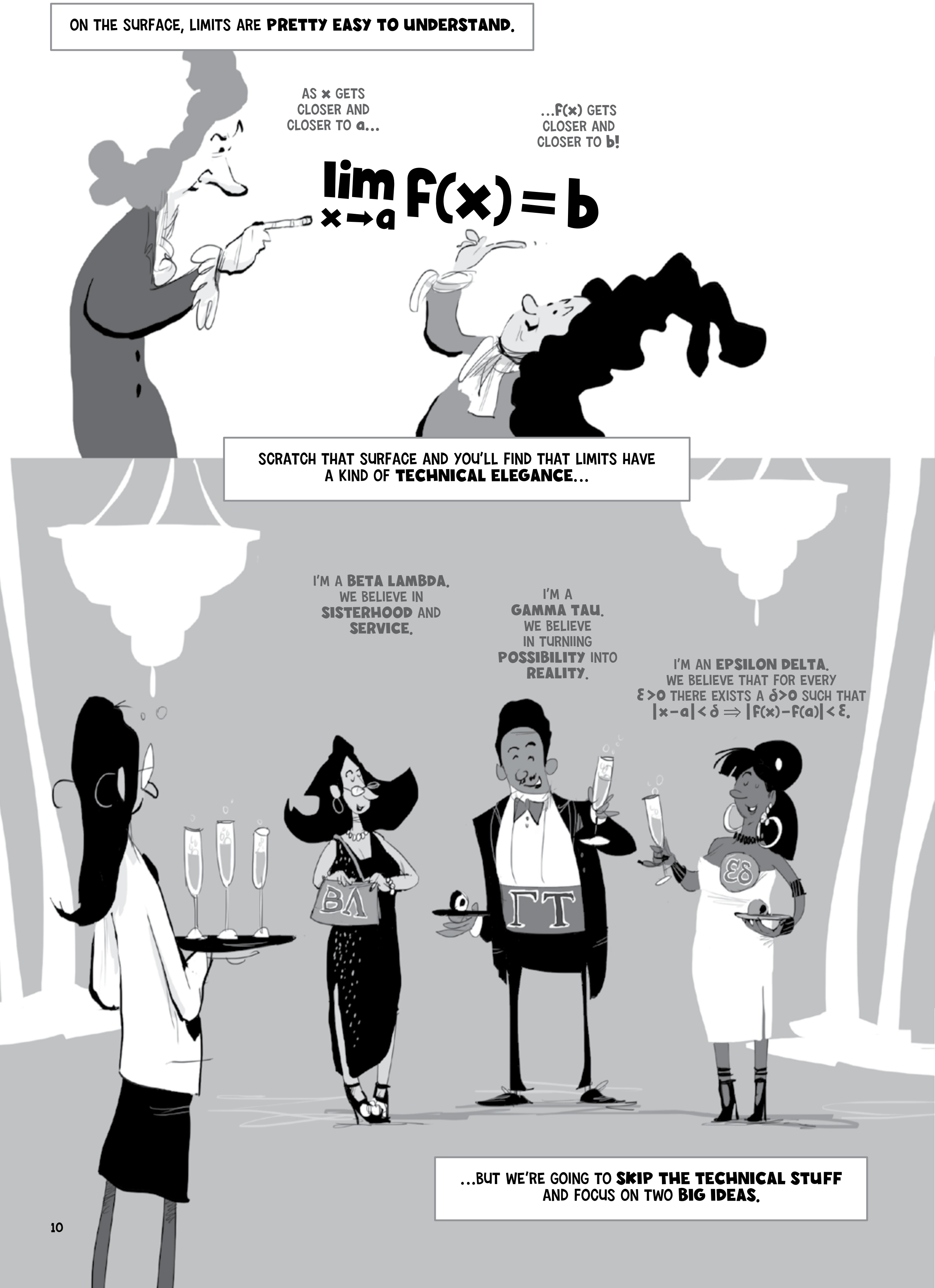

LIMITS ARE **NOT** ABOUT WHERE YOU'VE **BEEN**...
BYE BYE, **TEMPLE OF ATHENA.**
...OR WHERE YOU ARE **NOW.**
HELLO, **PARTHENON.**
LIMITS ARE ABOUT **WHERE YOU'RE GOING.**
THE LIMIT OF OUR TRIP IS THE **ORACLE AT DELPHI.**


THANKS TO LIMITS, WE CAN TAKE DICEY SITUATIONS...
WITH WHAT SPEED WILL POISON SHUFFLE OFF MY MORTAL COIL?
...WHICH SEEM TO GET
CLOSER AND CLOSER TO AN INEVITABLE CONCLUSION...
FAREWELL, O WRETCHED STATE!
ADIEU!
WHAT UGLY SIGHTS OF DEATH WITHIN MINE EYES!
THUS DIE I, THUS, THUS, THUS.
–COUGH–
–COUGH–
ADIEU...
GASP
ADIEU...
ADIEU...
ADIEU...

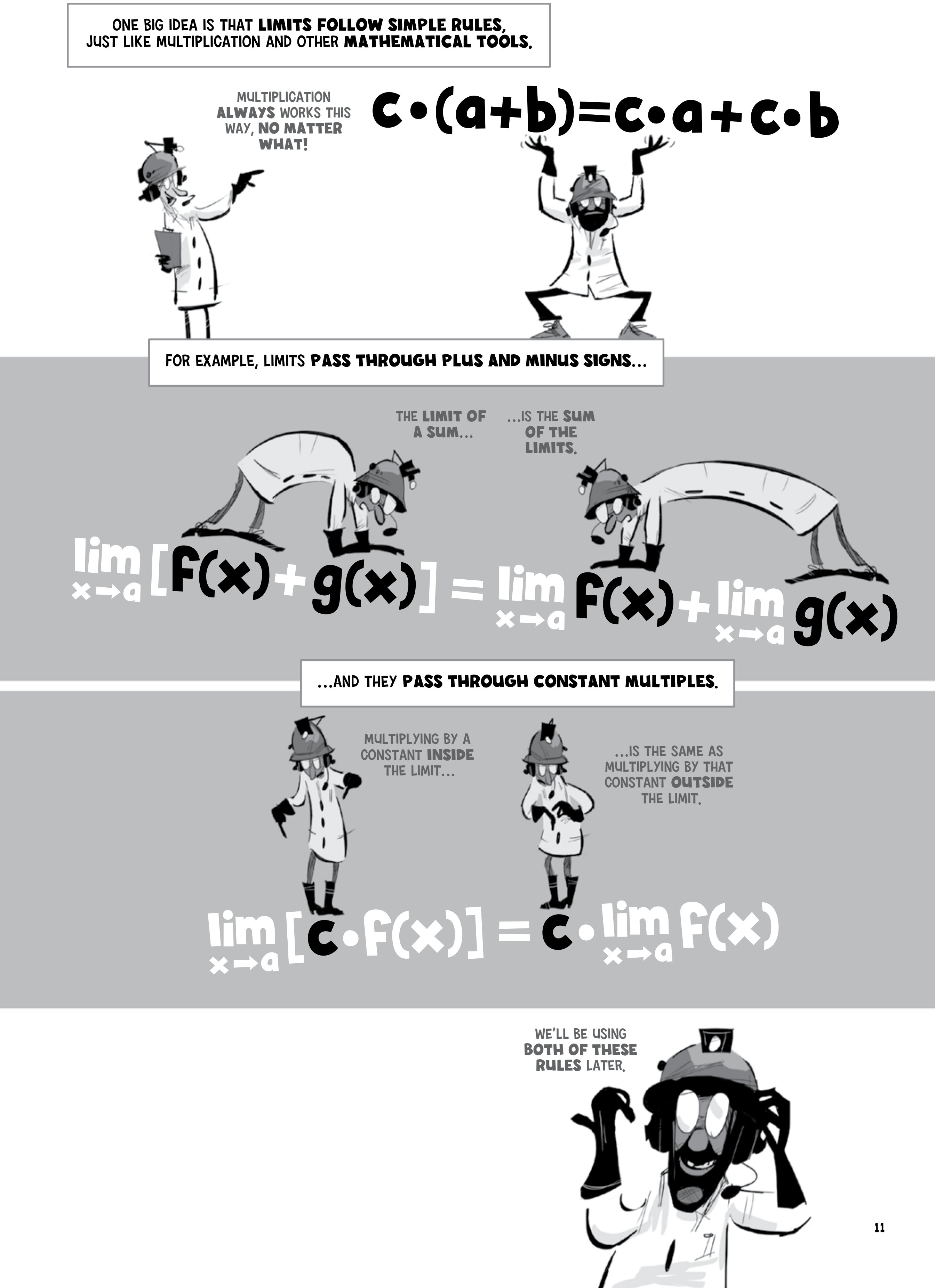

...AND FIND **CLOSURE.**
IN THE LIMIT, HE'S **DEAD.**
BRAVO!


FOR AN **EASY EXAMPLE**, CONSIDER THIS INFINITE SEQUENCE: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128} \ldots$
FIRST YOU GO HALFWAY FROM ZERO TO ONE, THAT'S $\frac{1}{2}$...
...THEN YOU GO HALFWAY AGAIN, THAT'S $\frac{3}{4}$...
...THEN YOU GO HALFWAY AGAIN, THAT'S $\frac{7}{8}$...
0
1/2
3/4
7/8
1
IT'S JUST LIKE **ZENO'S PARADOX**.
IT SHOULD MAKE INTUITIVE SENSE THAT THIS SEQUENCE HAS A **LIMIT OF 1.**
THAT'S **WHERE WE'RE GOING!**

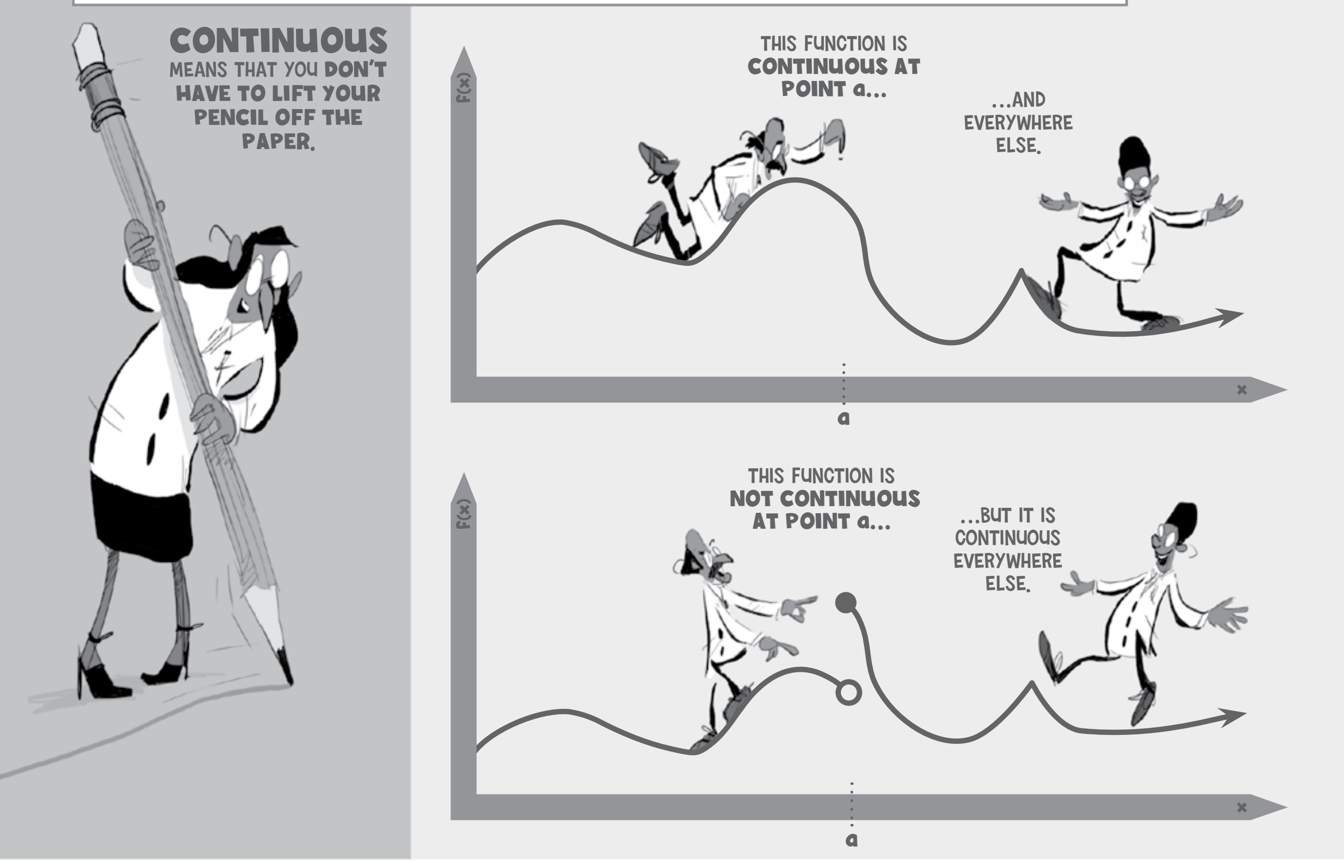

MATHEMATICALLY SPEAKING, THE SEQUENCE HAS A **LIMIT OF 1** BECAUSE EVENTUALLY IT GETS **ARBITRARILY CLOSE TO 1.**
DOES IT EVENTUALLY GET **WITHIN 0.1 OF 1?**
15/16
YES, AFTER THE **4th** ELEMENT OF THE SEQUENCE.
0
1/2
3/4
7/8
1
DOES IT EVENTUALLY GET **WITHIN 0.001 OF 1?**
1023/1024
YES, AFTER THE **10th** ELEMENT OF THE SEQUENCE.
0
1/2
3/4
7/8
1
DOES IT EVENTUALLY GET WITHIN **ε OF 1** FOR ANY **ε>0?**
YES, I GUARANTEE IT!
0
1/2
3/4
7/8
1
$\lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1$
THAT'S WHAT **THIS** MEANS.
THE SYMBOL ∞ STANDS FOR **INFINITY.**

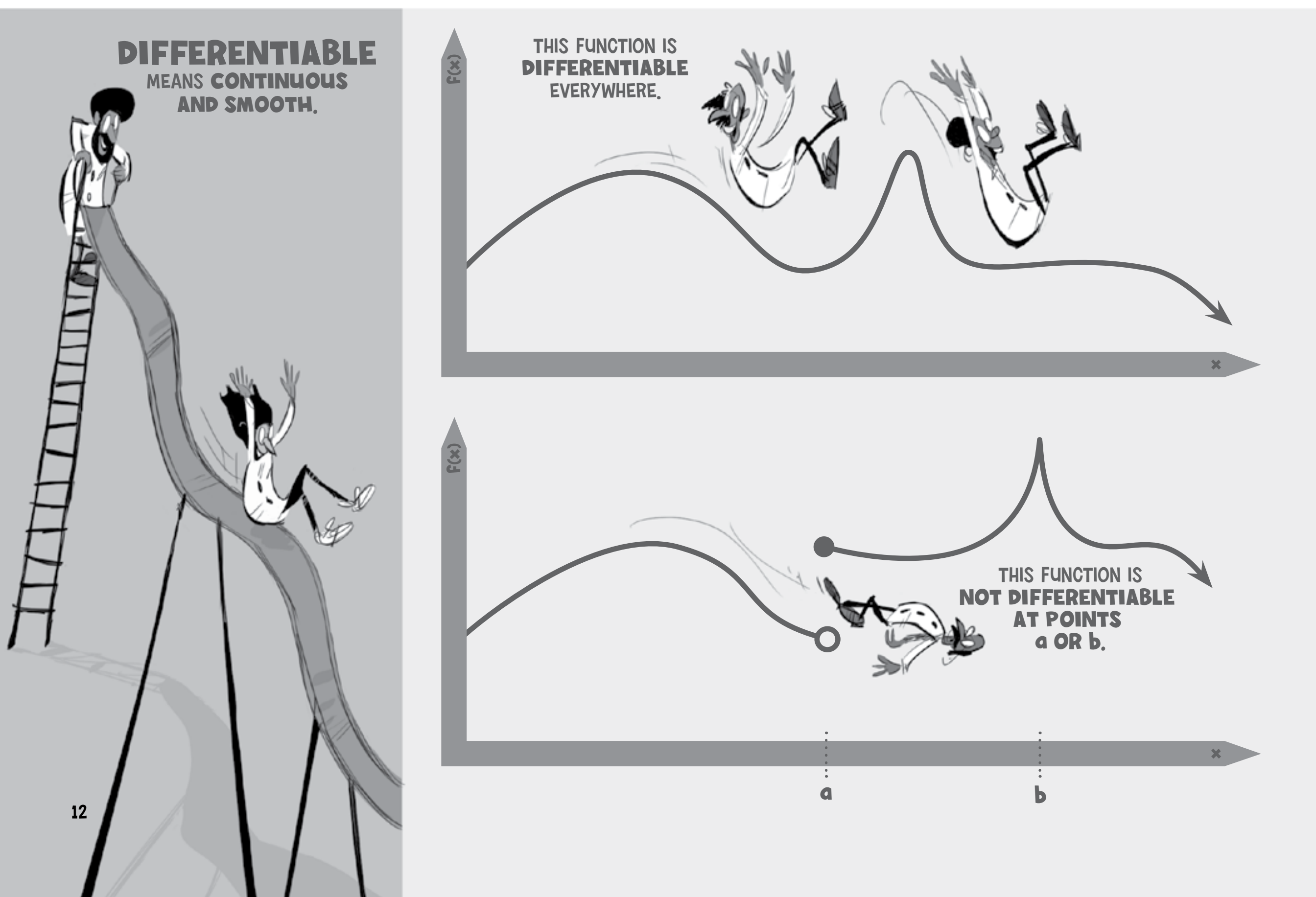


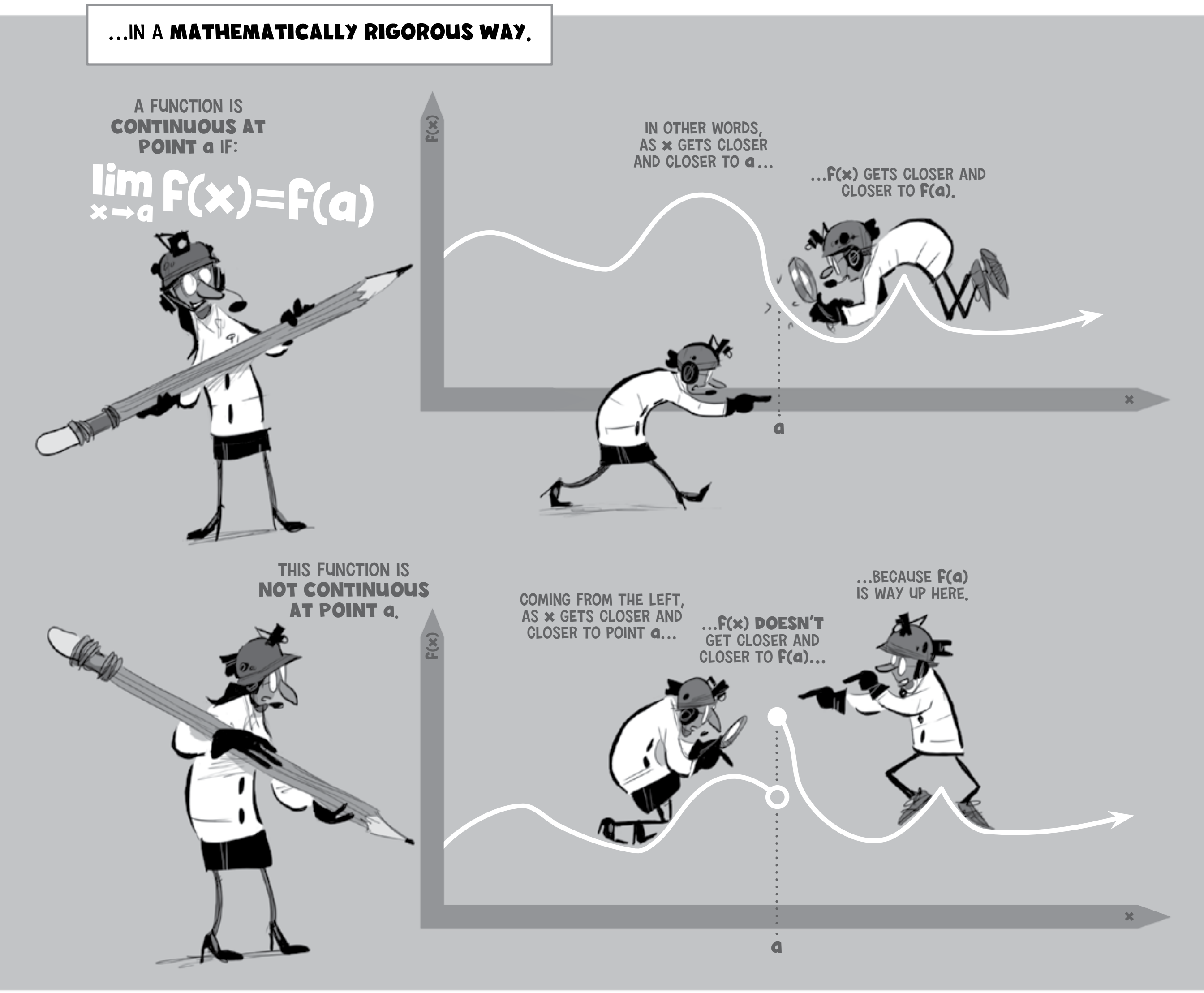
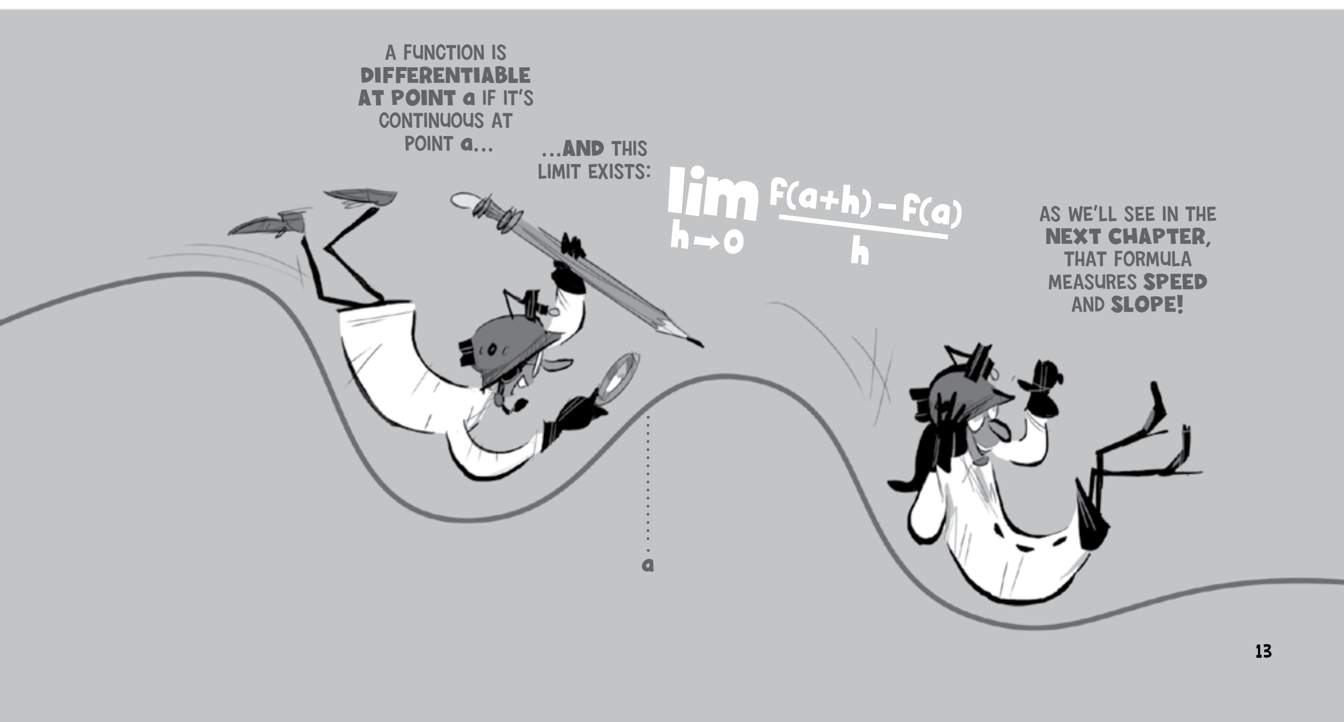
ON THE SURFACE, LIMITS ARE **PRETTY EASY TO UNDERSTAND.**
AS x GETS CLOSER AND CLOSER TO a...
...f(x) GETS CLOSER AND CLOSER TO b!
$\lim_{x \to a} f(x) = b$
SCRATCH THAT SURFACE AND YOU'LL FIND THAT LIMITS HAVE A KIND OF **TECHNICAL ELEGANCE**...
I'M A **BETA LAMBDA.** WE BELIEVE IN **SISTERHOOD** AND **SERVICE.**
I'M A **GAMMA TAU.** WE BELIEVE IN TURNIING **POSSIBILITY** INTO **REALITY.**
I'M AN **EPSILON DELTA.** WE BELIEVE THAT FOR EVERY $\varepsilon > 0$ THERE EXISTS A $\delta > 0$ SUCH THAT $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$.
ΒΛ
ΓΤ
ΕΔ
...BUT WE'RE GOING TO **SKIP THE TECHNICAL STUFF** AND FOCUS ON TWO **BIG IDEAS.**

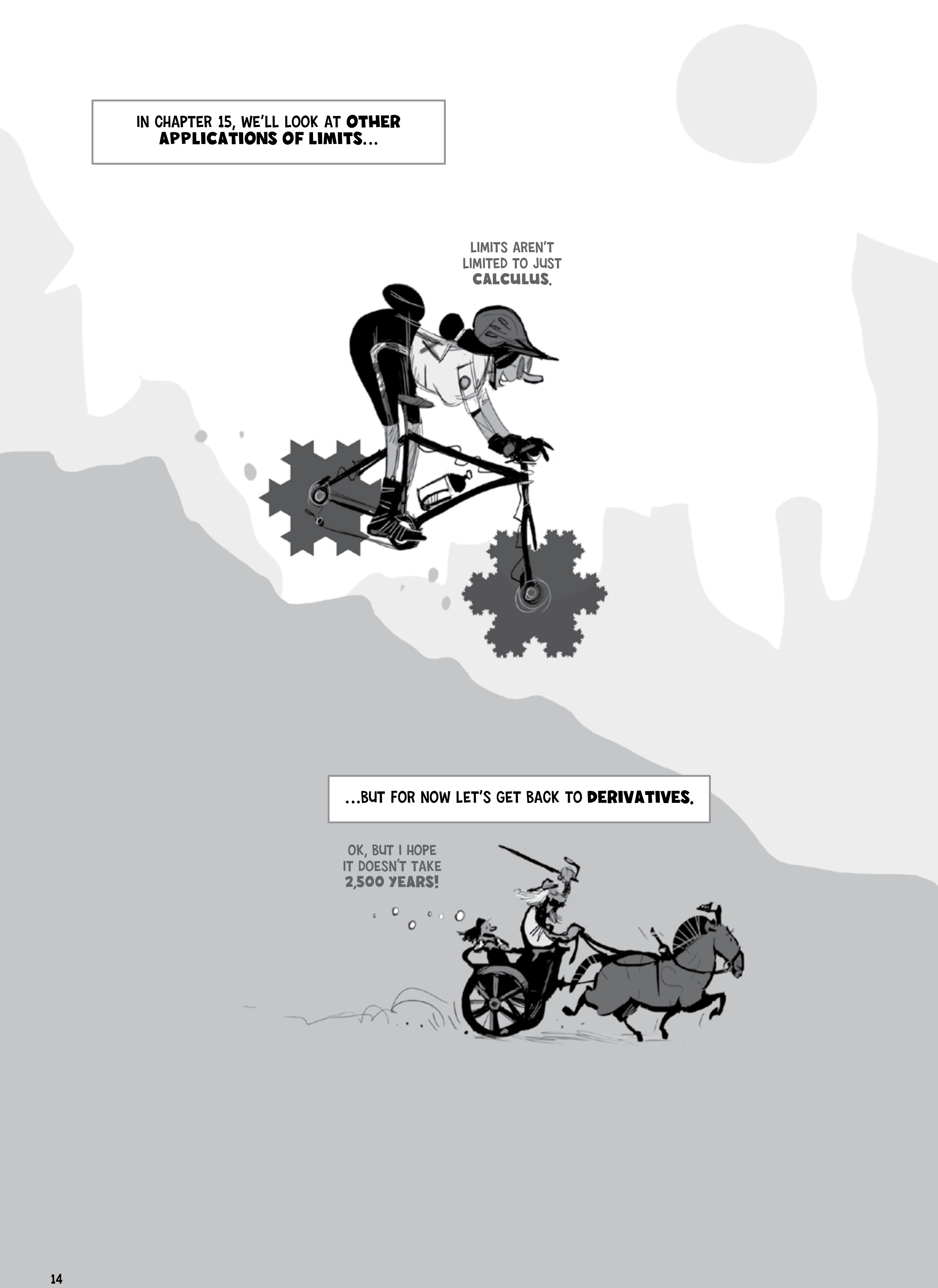

ONE BIG IDEA IS THAT **LIMITS FOLLOW SIMPLE RULES,** JUST LIKE MULTIPLICATION AND OTHER **MATHEMATICAL TOOLS.**
MULTIPLICATION **ALWAYS** WORKS THIS WAY, **NO MATTER WHAT!**
$c \cdot (a+b) = c \cdot a + c \cdot b$
FOR EXAMPLE, LIMITS **PASS THROUGH PLUS AND MINUS SIGNS...**
THE **LIMIT OF A SUM...**
...IS THE **SUM OF THE LIMITS.**
$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
...AND THEY **PASS THROUGH CONSTANT MULTIPLES.**
MULTIPLYING BY A CONSTANT **INSIDE** THE LIMIT...
...IS THE SAME AS MULTIPLYING BY THAT CONSTANT **OUTSIDE** THE LIMIT.
$\lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$
WE'LL BE USING **BOTH OF THESE RULES** LATER.

ANOTHER BIG IDEA IS THAT LIMITS MAKE IT POSSIBLE TO DESCRIBE **SIMPLE-SOUNDING CONCEPTS...**






...IN A **MATHEMATICALLY RIGOROUS WAY.**
A FUNCTION IS **CONTINUOUS AT POINT a** IF:
$$\lim_{x \to a} f(x) = f(a)$$
f(x)
IN OTHER WORDS, AS x GETS CLOSER AND CLOSER TO a...
...f(x) GETS CLOSER AND CLOSER TO f(a).
x
a
THIS FUNCTION IS **NOT CONTINUOUS AT POINT a.**
f(x)
COMING FROM THE LEFT, AS x GETS CLOSER AND CLOSER TO POINT a...
...f(x) **DOESN'T** GET CLOSER AND CLOSER TO f(a)...
...BECAUSE f(a) IS WAY UP HERE.
x
a

A FUNCTION IS **DIFFERENTIABLE AT POINT a** IF IT'S CONTINUOUS AT POINT a...
...**AND** THIS LIMIT EXISTS:
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
AS WE'LL SEE IN THE **NEXT CHAPTER,** THAT FORMULA MEASURES **SPEED** AND **SLOPE!**
a


IN CHAPTER 15, WE'LL LOOK AT OTHER APPLICATIONS OF LIMITS...
LIMITS AREN'T LIMITED TO JUST CALCULUS.
...BUT FOR NOW LET'S GET BACK TO DERIVATIVES.
OK, BUT I HOPE IT DOESN'T TAKE 2,500 YEARS!

THE INTERNATIONALLY BESTSELLING AUTHORS OF THE CARTOON INTRODUCTION TO ECONOMICS RETURN TO MAKE CALCULUS FUN

"*The Cartoon Introduction to Calculus* is hilarious, rigorous, slightly hallucinatory, and extremely educational, all at once—highly recommended for those who already love calculus and those encountering it for the first time."
—**JORDAN ELLENBERG**, John D. MacArthur Professor of Mathematics, University of Wisconsin–Madison, and author of *How Not to Be Wrong*

With Grady Klein and Yoram Bauman as our guides, we scale the dual peaks of Mount Derivative and Mount Integral, and from their summits we see how calculus relates to the rest of mathematics. Beginning with the problems of speed and area, Klein and Bauman show how the discipline is unified by a fundamental theorem. We meet such geniuses as Archimedes, Liu Hui, and Bonaventura Cavalieri, who survived the slopes on intuition but prepared us for the avalanche-like dangers posed by mathematical rigor. Then we trek onward, scrambling through limits, extreme values, optimization, and integration, and learn how calculus can be applied to economics, physics, and so much more. Klein and Bauman round out the book with a handy glossary of symbols and terms so you won't mix up constants and constraints. With a witty and engaging narrative full of jokes and insights, *The Cartoon Introduction to Calculus* is an essential primer for students or for anyone who is curious about math.

"Even the most timid mathophobe who is willing to dip a toe in the water will be enlightened and entertained by *The Cartoon Introduction to Calculus*. Kudos to Grady Klein and Yoram Bauman!"
—**MICHAEL SULLIVAN**, Chicago State University, and **KATHLEEN MIRANDA**, SUNY Old Westbury, authors of *Calculus for the AP Course*

GRADY KLEIN is a cartoonist, an animator, and a graphic designer who specializes in simplifying complex subjects. His most recent book is *Psychology: The Comic Book Introduction*. Samples of his work can be found at www.gradyklein.com.

YORAM BAUMAN, Ph.D., is an economist who performs at universities and corporate events around the world as "the world's first and only stand-up economist." His website is www.standupeconomist.com.

Klein and Bauman have previously collaborated on *The Cartoon Introduction to Climate Change*, the two-volume *Cartoon Introduction to Economics*, and, most recently, *The Cartoon Introduction to Digital Ethics*.

Cover art and design by Grady Klein

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